METAPHORS IN TEACHING AND LEARNING MATHEMATICS: A CASE STUDY CONCERNING INEQUALITIES

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ABSTRACT: In this paper an embodied cognition perspective is considered in order to frame teaching and learning problems concerning inequalities. The nature and functions of some "grounding metaphors" are discussed, as well as the possibility of enhancing their use by students.

1. Introduction

Since the beginning of the eighties metaphors have been reconsidered as crucial components of thinking (see Lakoff and Johnson, 1980). The relevance of body-related metaphors in mathematical thinking has been clearly stated by Lakoff and Nunez (1997) and Nunez et al (1999) with some examples concerning, in particular, natural numbers and continuity.

Nunez (2000) describes *conceptual metaphors* as follows: *"It is important to keep in mind that conceptual metaphors are not mere figures of speech, and that they are not just pedagogical tools used to illustrate some educational material. Conceptual metaphors are in fact fundamental cognitive mechanisms (technically, they are inference-preserving cross-domain mappings) which project the inferential structure of a source domain onto a target domain, allowing the use of effortless species-specific body-based inference to structure abstract inference"*. Considering *conceptual metaphors*, Lakoff and Nunez (2000) (see also Nunez, 2000) make a distinction between *grounding metaphors* (i. e. conceptual metaphors which *"ground our understanding of mathematical ideas in terms of everyday experience"*) and other kinds of conceptual metaphors (*Redefinitional metaphors, Linking metaphors*).

This paper has three aims:

- to show how different kinds of *grounding metaphors* can intervene (as crucial tools of thinking) in novices' approach to inequalities;

- to discuss possible refinements of the idea of a *grounding metaphor* deriving from the analysis of students' behaviour and related to the cultural variety of possible everyday life source domains;

- to investigate how *grounding metaphors* can become a legitimate tool of thinking for students.

In order to fulfil these aims, we will consider a teaching experiment performed in two VIII-grade classes (13-14 years old students) with the main purpose of detecting the young students' potential in dealing with inequalities approached in a functional way (i.e. by considering them as comparisons of functions) . We will consider also four Ph. D. students' behaviour in similar tasks.

2. Grounding Metaphors and Communication Metaphors

In mathematics as well as in other domains, many metaphors are frequently used with pure communication purposes: we will call them *communication metaphors*. For instance, in a sentence like *"The proof of this theorem was like an obstacle race: once some progress was made, immediately another difficulty appeared"* the *obstacle race* metaphor fulfils a pure communication purpose which has no mathematical content. *Communication metaphors* can also be employed in order to substitute for technical espressions which are not shared by the interlocutor: this happens frequently in a popularisation situation.

Can metaphors fulfil other functions, in particular thinking tool functions?

The examples produced in recent papers by Lakoff and Nunez (1997; 2000), Nunez, Edwards and Matos (1999) and Nunez (2000) show how some metaphors (*conceptual metaphors*) can function as thinking tools, in particular as ways of thinking about peculiar mathematical objects; these authors go further in hypothesizing that conceptual metaphors (in particular, *grounding metaphors* - i.e. those referring to everyday body actions and relations) are not exceptions, but usual ways of thinking in mathematics.

The distinction between *communication metaphors* and *grounding metaphors*, and the very existence of grounding metaphors in a given mathematics domain, are relevant for educational purposes: indeed communication metaphors could not solve deep learning problems and so they would demand weak engagement by teachers. On the contrary, teachers should legitimate and enhance the use of grounding metaphors as tools of thinking. Special attention should be paid to the existence and functions of grounding metaphors in those mathematics domains where difficulties of learning are greater: indeed ignoring specific grounding metaphors could be one of the reasons for students' difficulties (see the analysis of the difficulties inherent in the learning of continuity, in Nunez, Edwards and Matos, 1999).

In this paper we deal with grounding metaphors in the domain of inequalities; as we will see in the next section, for different reasons (including students' difficulties) this subject is poorly covered in current teaching. The challenge coming from an *embodied cognition* perspective (Lakoff and Nunez, 1997; Nunez, 2000) is to ascertain at what extent grounding metaphors can support the teaching and learning of this subject and overcome some of the students' difficulties.

Another problem concerns the necessity of establishing whether novices' *grounding metaphors* do substitute more advanced thinking tools; in this case their importance should be only temporary in the students' career. For this reason in our experimental study we considered both novices (VIII-graders) and university students (in particular, four Ph.D. students in mathematics) engaged in structurally similar tasks (even if the level of difficulty was different: see Annexes).

3. Rationale of the Experimental Study

In spite of their importance in pure and applied mathematics, inequalities constitute a neglected and ill-treated subject in secondary school curricula. In most countries, inequalities are taught as a subordinate subject (in relationship with equations), dealt with in purely algorithmic manner which avoids, in particular, the difficulties inherent in the concept of function (see Assude, 2000; Sackur and Maurel, 2000). As a consequence, students are unable to manage inequalities which do not fit the learned schemas. For instance, according to different independent studies (cfr. Boero, 2000; Malara, 2000) at the entrance of the university mathematics courses in Italy most students fail to solve easy non standard inequalities like $x^2 - 1/x > 0$.

Concerning the mathematical relevance of the current teaching of inequalities in school, we may observe that it does not take into account the importance of inequalities in pure and applied mathematics (for instance in the case of the concept of limit or to deal with asymptotic stability) and the fact that in many cases equations are solved with approximation methods which are based on inequalities (thus reversing the usual approach to inequalities in school - as a subject subordinate to equations).

This brief presentation brings to the following conclusion: the prevailing manner of teaching inequalities in school neither is efficient (as concerns the results, in terms of the capacity to deal with a large set of rather simple but non standard inequalities) nor fits relevant aspects of the professional (mathematicians') practice about inequalities. In order to try to find the reasons of this situation and elaborate tools for overcoming it, we can observe that the functional aspect plays a crucial role in mathematicians' work, both for equations and inequalities. This fact is often neglected in traditional teaching. From a functional point of view, inequalities fully involve difficult concepts like *variable* and *function* in situations which need a complex treatment. As suggested at the beginning of this section, we can recognize that the traditional teaching of inequalities avoids the "function" concept and reduces the difficulties ineherent in the *variable* concept and the complexity of the solution process by treating inequalities as a special case of equations.

Keeping in mind this analysis, when planning the teaching experiment with two VIII-grade classes our basic cultural choice consisted in treating equations and inequalities from a functional point of view, i.e. approaching them as special cases of comparison of functions.

Here some details about the classes, educational choices and classroom activities are reported:

- 36 VIII-grade students (divided into two classes) were involved; as usual in Italy, they had started to work with the same mathematics and science teacher in grade VI;

- the didactic contract established in grades VI, VII and at the beginning of grade VIII was coherent with the methodological choice of a cooperative, participated, guided enrichment of tools and skills in the planned activity. A rather common routine of classroom work consisted in individual production of written solutions for a given task (if necessary, supported by the teacher with 1-1 interventions), followed by classroom comparison and discussion of students' products, guided by the teacher and, possibly, by the adoption of other students' solutions in similar tasks. Another aspect of the didactic contract included the exhaustive written wording of doubts, discoveries, heuristics, etc.;

- the approach to the concept of function was built up through activities involving tables, graphs and formulas and mainly concerning geometric entities (lengths, perimeters, areas, etc.). After some initial activities on functions as machines (*operational view*) and then as x to y correspondences (*correspondence view:* see Slavit, 1997 for a survey about these different *views*), point-by-point drawing of graphs was discouraged, while making hypotheses about their shape (starting from their formula) was greatly encouraged;

- the role of the teacher was to help students through 1-1 interactions and manage students' classroom discussions. In particular, students were encouraged to communicate their ideas with words, gestures, graphs drawn at the blackboard, etc..

Collected materials from the two classes consisted in individual protocols, audio-recordings of classroom discussions and detailed teachers' notes. The same materials were collected in the case of the four Ph. D. students in Mathematics involved in this study for comparison purposes (see end of Section 2).

4. Grounding Metaphors and Inequalities

In the Annexe 1 some excerpts of a student's solution are reported; they are representative of a large set of protocols deriving from the written solutions of the 36 VIII-grade students considered in this study. Also a solution from a Ph. D. is reported (Annexe 2), to show impressive similarities between the novices' strategies in dealing with open problems concerning inequalities and the efforts of an expert young mathematician in dealing with a similar, more difficult task (in fact, a task not covered by learned procedures).

Different metaphors surface in students' protocols. For space restrictions we will consider only one crucial step of the solution: the search for *pivot points* around which the direction of the inequality changes. In many protocols we find one (or more) of the following metaphors:

- dynamical reference to increasing and decreasing values, and the necessity of a meeting point supported by consecutive dynamical gestures of one hand (firstly indicating increase, then decrease, or vice-versa); words are coherent with this body dynamic representation: students speak about *going up* and *going down* of the two functions (*formulas, graphs,* etc.), and thus they must "*meet in one point"* (*meeting metaphor*): *"one graph goes up steeper and steeper from below and must meet the other which increases and then goes down"* (see Annexe 2).

- reference to the imagined (or drawn) shapes of the two graphs, and the necessity of a meeting point supported by static crossing of the two arms; again words are coherent with this static body representation: the two graphs *"must have one point in common"* (*intersection metaphor*);

- *balance metaphor*: in this case the idea of a possible equilibrium between the values of the two functions drives the student's attention towards values of x which are near to satisfy the equation. A reference to physical trials performed in order to reach the equilibrium point is evident. We may note that the balance metaphor was used by students to guide the search for the equilibrium point in different ways: in particular with numerical trials on the two sides in order to approach the equilibrium point; or through a regular movement from left to right (see Annexe 1 for an example).

5. Discussion

In our opinion, the reported excerpts and the examples of the metaphors surfacing in VIII-grade students' attempts to find the 'pivot points' (as well as in the Ph. D. students protocols: see Annexes) raise three relevant questions:

Can we speak of grounding metaphors?

The communication function does not seem to be the most important function in the students' protocols: metaphors *"project the inferential structure of a source domain onto a target domain"* , according to Nunez's description of conceptual metaphors (see Section 1), and the relationships established in the different source domains (for instance: balance equilibrium) serve as crucial references to infer conclusions in the target domain (functions and inequalities). In particular, the necessity of a point belonging to both graphs derives from the necessity that can be experienced in the source domain (see later for a detailed analysis).

Can we consider the grounding metaphors used by students as spontaneous, or may we identify their origin in classroom activities?

The knowledge of students' background brings to the hypothesis that words and gestures (strongly encouraged by the teacher during the previous classroom activities on functions: see Section 3) allowed different kinds of grounding metaphors concerning functions and variables to become legitimate and spread in the classroom. We would like to make some comments about *legitimacy* and *spreading. Legitimacy* means that students were allowed to overtly reason through those kinds of grounding metaphors. This is not frequent in mathematics teaching (even in lower grades): abstract reasonings are privileged. *Spreading* means that overt, legitimate gestures and words were freely adopted by the schoolmates, according to their personal needs; by this way different grounding metaphors became accessible as thinking tools for dealing with variables and functions. For instance, the balance metaphor was produced by the author of the first protocol (Annexe 1) in a previous situation; the teacher promoted a discussion about it; then it spread in the classroom (9 students used it in the task reported in Annexe 1).

If this perspective is appropriate to describe what happened in the two classrooms, the teacher's role seems to be crucial in order to provide students with the opportunity of accessing powerful grounding metaphors. We can imagine that potentially every student can use them; but this potential does not translate into an effective, appropriate use if this use is not legitimate and supported by appropriate signs (particularly words and gestures). From the research point of view it would be interesting to better understand the specific role of signs (words and gestures) in the appropriation (or activation) and functioning of the grounding metaphors considered.

In the case of the search for pivot points how can we distinguish between the three kinds of metaphors described above?

The very nature of the source domains show important differences between the different metaphors. All of them enter the definition of *grounding metaphor* quoted in Section 1, but the nature of the evoked *everyday experience* is not the same in the different cases: again considering the three grounding metaphors surfacing in the research for the pivot points, in the first case we can recognize a reference to an everyday experience concerning crossing of movements. It is interesting to observe that the gestures for the description of this situation are the same that we would use in describing the crossing of two people climbing and descending a staircase. A physical experience concerning ordinary life supports the necessity of a *meeting point*. In the second case, a familiar situation of a necessary crossing of two continuous lines is evoked: the activity of drawing lines on plane surfaces provides the support for the visual necessity of a *common point*. In the third case, an everyday life technological situation is evoked: a technological tool (the balance) provides the physical necessity of an equilibrium point (in fact, the *pivot point*). These remarks suggest the consideration of different kinds of *everyday experience,* with different relationships between *culture* and *body*: an immediate relationship in the first case, a visual culture-mediated relationship in the second case, a technology-mediated relationship in the third case. In other words, we could speak of different culture-mediated necessities for a *pivot point* in the three cases.

Even these remarks suggest some educational implications: a variety of everyday experiences should be recognized and encouraged by the teacher in the classroom as legitimate sources of grounding metaphors for crucial mathematics concepts and situations.

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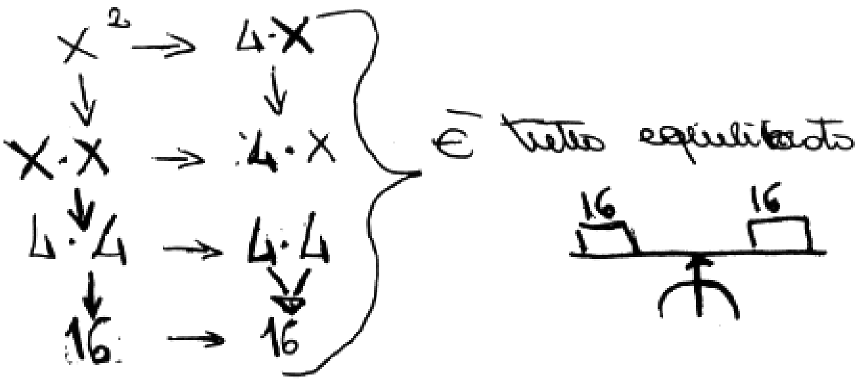
Annexe 1: some excerpts from the first part of an VIII-grade student's solution to the following problem:

Compare the following formulas from an algebraic and a graphical pont of view. Make hypotheses about their graphs and motivate them carefully, finally draw a sketch of their graphs. A) $y= x^2- 4x +4$ B) $y= -x^2+4$

"Due to the fact that x^2, that is x multiplied by itself, is there, we should get two parabolas. The presence of +4 makes them start from +4 when x=0

The first curve will meet twice the second: at +4 (on the y axis) and in another point to be found. The first curve will go down, in the first quadrant, below 4 when the weight of 4x will be greater than x^2 and will go up again when this situation will be over.

When does the first curve go below +4? Moving from 0 on the right, the first curve will go below +4 till when -4x will <u>*balance*</u> *x^2 , that is they will become* <u>*equal*</u>



The second parabola is a very usual parabola, but it is translated "upwards" by +4 and made negative as an effect of -x^2 which makes everything negative. It will remain over the level 0 for a while, until +4 will remain greater than -x^2. But when will x^2 equilibrate +4? (drawing: again a schematic representation of a balance). *When -x^2 will give a number that will annihilate +4, that is -4. When? -2·2+4=-4+4=0 [...]"*

Annexe 2: a Ph. D. student's solution for the following problem: *"To find where xsinx>x^2-1"*. The student is invited to tell aloud what he thinks.

(from audiorecording and notes taken by the interviewer)

"It is evident that the parabole overcomes the other function when x grows in absolute value, because xsinx cannot be bigger than the absolute value of x. It is a parabole compared with two straight lines outcoming from the origin and moving upwards (he makes gestures showing the two curves, then he makes a sketch on paper). *The problem is what happens near to zero. Indeed if x=0 I see that the parabole is below the other function. But... xsinx... it is a pair function, a symmetric function... Well, I can consider only the positive side of the x axis. Here I imagine... x^2-1 is like x^2 lowered by one* (gestures in the air: a parabole then a lower parabole). *OK, the other function goes up and down, but definitively it will remain below the parabole... I have already said it* (he points to the drawing). *Now I must coordinate what happens near to zero and what happens at large* (he carefully draws the parabole y=x^2-1, the y=-x and the y=x straight lines). *I must be more precise, and see where xsinx meets the x axis* (he makes a sign for 1, 2,3,4, 5, 6, then he makes a sketch of the graph of xsinx for x>0, saying: *"it goes up and down between these two straight lines"*). *It looks fine. Oh, oh, this sketch is not precise enough - I must find where xsinx meets x... OK, sinx=1, it is here (he makes a sign on the straight line by going up from the value of approx. 1.5 on the x axis, then he draws a more precise graph of y=xsinx between 0 and 3). xsinx=x^2-1 ... no precise solution, but a solution do exist, I see here, one graph goes up steeper and steeper from below and must meet the other which increases and then goes down. But I should get no more than one solution... Let us see: if x=1.5, x^2-1 makes 2.25-1, that is 1.25... bigger than one, but not so bigger... It means that the meeting point is near to 1.5 on the left... OK, my drawing was OK! I can see that x^2-1 overcomes xsinx out of this interval* (he rapidly completes the drawing by symmetry on the left, and makes

symmetric gestures with the two hands to indicate the two symmetric parts of the x axis, out of the central interval)".